

MEAN VALUE THEOREMS FOR BINARY EGYPTIAN FRACTIONS II

JING-JING HUANG AND ROBERT C. VAUGHAN

ABSTRACT. In this article, we continue with our investigation of the Diophantine equation $\frac{a}{n} = \frac{1}{x} + \frac{1}{y}$ and in particular its number of solutions $R(n; a)$ for fixed a . We prove a couple of mean value theorems for the second moment $(R(n; a))^2$ and from which we deduce $\log R(n; a)$ satisfies a certain Gaussian distribution with mean $\log 3 \log \log n$ and variance $(\log 3)^2 \log \log n$, which is an analog of the classical theorem of Erdős and Kac. And finally these results in all suggest that the behavior of $R(n; a)$ resembles the divisor function $d(n^2)$ in various aspects.

1. INTRODUCTION

In the previous memoir of this series (see [HV]) we studied the mean value

$$S(N; a) = \sum_{\substack{n \leq N \\ (n, a) = 1}} R(n; a), \quad (1)$$

of the number $R(n; a)$ of positive integer solutions to the Diophantine equation

$$\frac{a}{n} = \frac{1}{x} + \frac{1}{y}. \quad (2)$$

Here we extend our investigation to the second moment and some consequences thereof.

Theorem 1. *For fixed positive integer a , we have, for every $N \in \mathbb{N}$ with $N \geq 2$,*

$$\sum_{\substack{n \leq N \\ (n, a) = 1}} \left| R(n; a) - \frac{1}{\phi(a)} \sum_{\substack{\chi \bmod a \\ \chi^2 = \chi_0}} \bar{\chi}(-n) \sum_{u|n^2} \chi(u) \right|^2 \ll_a N \log^2 N,$$

where \ll_a indicates that the implicit constant depends at most on a , and where χ_0 denotes the principal character modulo a .

In the character sum here the term $\chi = \chi_0$ contributes an amount $d(n^2)$ where d is the divisor function and we can expect that this is the dominant contribution on average. Thus as a consequence of the Erdős–Kac theorem, just as for the divisor function $d(n)$, one can anticipate that $\log R(n; a)$ has a Gaussian distribution. As a first approximation we establish the normal order of $\log R(n; a)$.

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Theorem 2. *When a is fixed, the normal order of $\log R(n; a)$ as a function of n is $(\log 3) \log \log n$.*

Let

$$\Phi(z) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^z e^{-\frac{t^2}{2}} dt.$$

Then with a little more work we can establish the full distribution.

Theorem 3. *For fixed positive integer a , we have*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \text{card} \left\{ n \leq N : \frac{\log R(n; a) - (\log 3) \log \log n}{(\log 3) \sqrt{\log \log n}} \leq z \right\} = \Phi(z).$$

For completeness we also establish the mean square of $R(n; a)$ for fixed a . Since $R(n; a)$ resembles quite closely the divisor function $d(n^2)$ in many aspects, we expect that their mean squares share the same order of magnitude. Thus the following theorem can be compared with the asymptotic formula

$$\sum_{n \leq N} d^2(n^2) = NP_8(\log N) + O(N^{1-\delta})$$

which holds for some $\delta > 0$ and with $P_8(\cdot)$ a polynomial of degree 8.

Theorem 4. *Let a be a fixed positive integer and $\varepsilon > 0$. Then*

$$\sum_{\substack{n \leq N \\ (n, a) = 1}} R(n; a)^2 = NP_8(\log N; a) + O_a(N^{35/54+\varepsilon})$$

where $P_8(\cdot; a)$ is a degree 8 polynomial with coefficients depending on a , and its leading coefficient is

$$\frac{1}{8!a^2} \prod_{p|a} \left(1 - \frac{1}{p}\right)^7 \prod_{p \nmid a} \left(1 - \frac{6}{p} + \frac{1}{p^2}\right) \left(1 - \frac{1}{p}\right)^6.$$

The error term in the theorem above is closely related to the generalised divisor problem, and in particular depends on a mean value estimate for the ninth moment of Dirichlet L -functions $L(s, \chi)$ modulo a inside the critical strip. As is easily verified, the error can be improved to $O_a(N^{1/2+\varepsilon})$ under the assumption of the generalised Lindelöf Hypothesis.

2. PROOF OF THEOREM 1

We rewrite equation (2) in the form

$$(ax - n)(ay - n) = n^2.$$

After the change of variables $u = ax - n$ and $v = ay - n$, it follows that $R(n; a)$ is the number of ordered pairs of natural numbers u, v such that $uv = n^2$ and $u \equiv v \equiv -n \pmod{a}$.

Under the assumption that $(n, a) = 1$, $R(n; a)$ can be reduced further to counting the number of divisors u of n^2 with $u \equiv -n \pmod{a}$. Now the residue class $u \equiv -n \pmod{a}$ is readily isolated *via* the orthogonality of the Dirichlet characters χ modulo a . Thus we have

$$R(n; a) = \frac{1}{\phi(a)} \sum_{\substack{\chi \\ \text{mod } a}} \bar{\chi}(-n) \sum_{u|n^2} \chi(u), \quad (3)$$

where the condition $(n, a) = 1$ is taken care of by the character $\bar{\chi}(n)$.

Hence

$$\begin{aligned}
& \sum_{\substack{n \leq N \\ (a, n) = 1}} \left| R(n; a) - \frac{1}{\phi(a)} \sum_{\substack{\chi \bmod a \\ \chi^2 = \chi_0}} \bar{\chi}(-n) \sum_{u|n^2} \chi(u) \right|^2 \\
& \ll_a \sum_{\substack{n=1 \\ (a, n) = 1}}^{\infty} e^{-n/N} \left| \sum_{\substack{\chi \bmod a \\ \chi^2 \neq \chi_0}} \bar{\chi}(-n) \sum_{u|n^2} \chi(u) \right|^2 \\
& = \sum_{\substack{\chi_1 \bmod a \\ \chi_1^2 \neq \chi_0}} \sum_{\substack{\chi_2 \bmod a \\ \chi_2^2 \neq \chi_0}} \bar{\chi}_1 \chi_2(-1) \sum_{n=1}^{\infty} \sum_{u|n^2} \sum_{v|n^2} \chi_1(u) \bar{\chi}_2(v) \bar{\chi}_1 \chi_2(n) e^{-n/N},
\end{aligned}$$

where χ_0 denotes the principal character modulo a . In order to evaluate the sum over n , we analyze the Dirichlet series

$$f_{\chi_1, \chi_2}(s) := \sum_{n=1}^{\infty} \sum_{u|n^2} \sum_{v|n^2} \chi_1(u) \bar{\chi}_2(v) \bar{\chi}_1 \chi_2(n) n^{-s}.$$

The condition $u|n^2$ can be written as $u_1 u_2^2 | n^2$ with u_1 squarefree, i.e. $u_1 u_2 | n$, and likewise for $v|n^2$. Thus

$$f_{\chi_1, \chi_2}(s) = \sum_{m=1}^{\infty} \frac{\bar{\chi}_1 \chi_2(m)}{m^s} \sum_{d=1}^{\infty} \frac{F(d)}{d^s} \quad (4)$$

where

$$F(d) = \sum_{\substack{u_1, u_2, v_1, v_2 \\ d = [u_1 u_2, v_1 v_2]}} \mu^2(u_1) \mu^2(v_1) \chi_1(u_1 u_2^2) \bar{\chi}_2(v_1 v_2^2) \bar{\chi}_1 \chi_2(d).$$

The function F is multiplicative and so the inner sum above is

$$\prod_p \left(1 + \sum_{k=1}^{\infty} F(p^k) p^{-ks} \right), \quad (5)$$

where

$$F(p^k) = \sum_{\substack{u_1, u_2, v_1, v_2 \\ [u_1 u_2, v_1 v_2] = p^k}} \mu^2(u_1) \mu^2(v_1) \chi_1(u_1 u_2^2) \bar{\chi}_2(v_1 v_2^2) \bar{\chi}_1 \chi_2(p^k). \quad (6)$$

In particular we have

$$F(p) = \chi_0(p) + \sum_{\chi \in \mathcal{X} \setminus \{\bar{\chi}_1 \chi_2\}} \chi(p),$$

where $\mathcal{X} = \{\chi_1, \chi_2, \chi_1 \chi_2, \chi_1 \bar{\chi}_2, \bar{\chi}_1, \bar{\chi}_2, \bar{\chi}_1 \bar{\chi}_2, \bar{\chi}_1 \chi_2\}$ (and the entries are considered to be formally distinct), and

$$|F(p^k)| \leq 8k.$$

Thus the Dirichlet series f converges absolutely for $\sigma > 1$ and

$$f_{\chi_1, \chi_2}(s) = G_{\chi_1, \chi_2}(s) L(s, \chi_0) \prod_{\chi \in \mathcal{X}} L(s, \chi), \quad (7)$$

where $G_{\chi_1, \chi_2}(s)$ is a function which is analytic in the region $\Re s > 1/2$ and satisfies

$$G(s) \ll 1 \quad (\sigma \geq \tfrac{1}{2} + \delta)$$

for any fixed $\delta > 0$. As χ_1, χ_2 are not characters of order 1 or 2, $f_{\chi_1, \chi_2}(s)$ has a triple pole at $s = 1$ when $\chi_1 = \chi_2$ or $\chi_1 \chi_2 = \chi_0$, and a simple pole otherwise. By Corollary 1.17 and Lemma 10.15 of [MV], for fixed a ,

$$L(s, \chi) - \frac{E(\chi)\phi(a)}{a(s-1)} \ll 2 + |t|$$

uniformly for $\sigma \geq \frac{1}{2}$ where $E(\chi)$ is 1 when $\chi = \chi_0$ and 0 otherwise. Hence by (5.25) of [MV]

$$\sum_{n=1}^{\infty} \sum_{u|n^2} \sum_{v|n^2} \chi_1(u) \bar{\chi}_2(v) \bar{\chi}_1 \chi_2(n) e^{-n/N} = \frac{1}{2\pi i} \int_{\theta-i\infty}^{\theta+i\infty} f_{\chi_1, \chi_2}(s) N^s \Gamma(s) ds,$$

where $\theta > 1$. Since the gamma function decays exponentially fast on any vertical line we may move the vertical path to the $\frac{3}{4}$ -line picking up the residue of the integrand at $s = 1$. The residue contributes an amount

$$\ll N(\log N)^2$$

and the new vertical path contributes

$$\ll N^{\frac{3}{4}}.$$

This completes the proof of Theorem 1.

3. PROOF OF THEOREM 2

By Theorem 1, we expect that for almost all n with $(a, n) = 1$, $R(n; a)$ is close to

$$\frac{1}{\phi(a)} \sum_{\substack{\chi \bmod a \\ \chi^2 = \chi_0}} \bar{\chi}(-n) \sum_{u|n^2} \chi(u).$$

Thus we need to examine the contribution from the characters modulo a of order 1 and 2. For general a , there may be many quadratic characters modulo a . Nevertheless we believe that the major contribution to the sum above comes from the principle character, and this is of size

$$\frac{d(n^2)}{\phi(a)}.$$

Thus, for fixed a , $\log R(n; a)$ should have the normal order of $\log d(n^2)$, namely $(\log 3) \log \log n$. When $(n, a) > 1$ we have

$$R(n; a) = R(n/(n, a); a/(n, a)) \tag{8}$$

and so we can expect that the general case follows from the special case $(n, a) = 1$.

Before embarking on the proof of Theorem 2, we state a lemma. We define, for any quadratic character χ ,

$$\Omega_{\chi}(n) = \text{card} \{p, k : k \geq 1, p^k | n, \chi(p^k) = 1\}.$$

Lemma 5. *Suppose that χ is a quadratic character to a fixed modulus a and that $N \geq 3$. Then*

$$\sum_{n \leq N} \left(\Omega_\chi(n) - \frac{1}{2} \log \log N \right)^2 \ll N \log \log N$$

and

$$\sum_{1 < n \leq N} \left(\Omega_\chi(n) - \frac{1}{2} \log \log n \right)^2 \ll N \log \log N.$$

Proof. The proof follows in the same way as Turán's theorem (see Theorem 2.12 of [MV]) on observing that

$$\sum_{\substack{p \leq N \\ \chi(p)=1}} \frac{1}{p} = \frac{1}{2} \log \log N + O(1)$$

and this is readily deduced from Corollary 11.18 of [MV]. \square

It is an immediate consequence of the above lemma that $\Omega_\chi(n)$ has normal order $\frac{1}{2} \log \log n$. In particular, for any fixed $\varepsilon > 0$, for almost all n ,

$$3^{\Omega_\chi(n)} < 3^{(\frac{1}{2} + \varepsilon) \log \log n}.$$

Now, for any quadratic character χ modulo a , let

$$g_\chi(n) = \sum_{u|n^2} \chi(u).$$

This is

$$\prod_{p^k \| n} (1 + \chi(p) + \chi^2(p) + \cdots + \chi^{2k}(p)).$$

When $\chi(p) = -1$ the general factor is 1, and when $\chi(p) = 1$ it is $2k + 1$. Hence

$$0 < g_\chi(n) \leq 3^{\Omega_\chi(n)}.$$

Thus for any fixed $\varepsilon > 0$, for every quadratic character modulo a , for almost all n ,

$$g_\chi(n) < (\log n)^{(\frac{1}{2} \log 3 + \varepsilon)}. \quad (9)$$

Let

$$r(n; a) = \frac{1}{\phi(a/(n, a))} \sum_{\substack{\chi \bmod a/(n, a) \\ \chi^2 = \chi_0}} \bar{\chi}(-n/(n, a)) g_\chi(n/(n, a)).$$

Since $R(n; a) = R(n/(n, a); a/(n, a))$, it follows by Theorem 1 that

$$\sum_{n \leq N} (R(n; a) - r(n; a))^2 = \sum_{d|a} \sum_{\substack{m \leq N/d \\ (m, a/d)=1}} (R(m; a/d) - r(m; a/d))^2 \ll N(\log N)^2.$$

Hence, for any fixed $\varepsilon > 0$, for almost all n we have

$$|R(n; a) - r(n; a)| < (\log n)^{1+\varepsilon}.$$

Therefore, by (9), for almost all n ,

$$\left| R(n; a) - \frac{d((n/(a, n))^2)}{\phi(a/(a, n))} \right| < (\log n)^{1+2\varepsilon}. \quad (10)$$

Now $3 \leq d(p^{2k}) = 1 + 2k \leq 3^k$. Hence

$$3^{\omega(n)-\omega(a)} \leq d((n/(a,n))^2) \leq 3^{\Omega(n)} \quad (11)$$

and it follows that

$$(\log n)^{\log 3 - \varepsilon} < \frac{d((n/(a,n))^2)}{\phi(a/(a,n))} < (\log n)^{\log 3 + \varepsilon}$$

for almost all n . Theorem 2 now follows.

4. PROOF OF THEOREM 3

By (10) and (11), for every fixed $\varepsilon > 0$, for almost all n ,

$$\frac{3^{\omega(n)}}{\phi(a/(a,n))} - (\log n)^{1+\varepsilon} < R(n; a) < 3^{\Omega(n)} + (\log n)^{1+\varepsilon}.$$

Moreover, for almost all n we have $\Omega(n) \geq \omega(n) > (1 - \varepsilon) \log \log n$. Hence for any δ with $0 < \delta < \log 3 - 1$ we have, for almost all n

$$3^{\omega(n)-\omega(a)-\log \phi(a/(a,n))} \exp(-(\log n)^{-\delta}) < R(n; a) < 3^{\Omega(n)} \exp((\log n)^{-\delta})$$

and so

$$3^{\omega(n)} \exp(-\varepsilon \sqrt{\log \log n}) < R(n; a) < 3^{\Omega(n)} \exp(\varepsilon \sqrt{\log \log n}).$$

Let

$$S(N; z) = \text{card} \left\{ n \leq N : \frac{\log R(n; a) - (\log 3) \log \log n}{\log 3 \sqrt{\log \log n}} \leq z \right\},$$

$$S_-(N; z) = \text{card} \left\{ n \leq N : \frac{\Omega(n) - \log \log n}{\sqrt{\log \log n}} \leq z \right\}$$

and

$$S_+(N; z) = \text{card} \left\{ n \leq N : \frac{\omega(n) - \log \log n}{\sqrt{\log \log n}} \leq z \right\},$$

Then for a non-negative monotonic function $\eta(n)$ tending to 0 sufficiently slowly as $N \rightarrow \infty$ we have

$$-\eta(N)N + S_-(N; z - \varepsilon) < S(N; z) < \eta(N)N + S_+(N; z + \varepsilon).$$

Hence, by the Erdős-Kac theorem (see, for example Theorem 7.21 and Exercise 7.4.4 of [MV]),

$$\Phi(z - \varepsilon) \leq \liminf_{N \rightarrow \infty} N^{-1} S(N; z) \leq \limsup_{N \rightarrow \infty} N^{-1} S(N; z) \leq \Phi(z + \varepsilon).$$

The theorem now follows from the continuity of Φ .

5. PROOF OF THEOREM 4

By a similar discussion to that in §2, we can show that the generating Dirichlet series for $R(n; a)^2$ is

$$\sum_{\substack{n=1 \\ (n,a)=1}}^{\infty} \frac{R(n; a)^2}{n^s} = \frac{1}{\phi(a)^2} \sum_{\substack{\chi_1, \chi_2 \\ \text{mod } a}} \bar{\chi}_1 \chi_2(-1) f_{\chi_1, \chi_2}(s),$$

where $f_{\chi_1, \chi_2}(s)$ is analytic in the region $\Re s > 1/2$ and is given by (7). Here $f_{\chi_1, \chi_2}(s)$ has a pole at 1 of order at least 1, and as high as 9 exactly when χ_1 and

χ_2 are equal to the principle character χ_0 . Now on applying Perron's formula, we have for $\theta = 1 = 1 + 1/\log(2N)$,

$$\sum_{\substack{n \leq N \\ (n,a)=1}} R(n;a)^2 = \frac{1}{\phi(a)^2} \sum_{\substack{\chi_1, \chi_2 \\ \text{mod } a}} \frac{\bar{\chi}_1 \chi_2(-1)}{2\pi i} \int_{\theta-iT}^{\theta+iT} f_{\chi_1, \chi_2}(s) \frac{N^s}{s} ds + O_a(N^{1+\varepsilon}/T). \quad (12)$$

Since we are shooting for the asymptotics for the the mean square, smoothing factors of the kind used in section 2 are best avoided. Since the integrand includes a product of nine L -functions, we cannot expect to be able to move the vertical integral path too close to the $1/2$ -line in the current state of knowledge. Nevertheless, the following result of Meurman [Me1] which extends Heath-Brown's theorem [H-B] on the twelfth power moment of the Riemann zeta function to Dirichlet L -functions, provides a starting point for the analysis.

Lemma 6.

$$\sum_{\chi \text{ mod } a} \int_{-T}^T |L(\tfrac{1}{2} + it, \chi)|^{12} dt \ll a^3 T^{2+\varepsilon},$$

where $\varepsilon > 0$, $a \geq 1$ and $T \geq 2$.

Then, adapting the argument of Chapter 8 of Ivić [Iv] for the Riemann zeta function to Dirichlet L -functions establishes the following.

Lemma 7.

$$\int_{-T}^T |L(\tfrac{35}{54} + it, \chi)|^9 dt \ll_a T^{1+\varepsilon},$$

where $\varepsilon > 0$, χ is a fixed Dirichlet character modulo $a \geq 1$ and $T \geq 2$.

If one utilizes the sharpest estimates for the underlying exponential sums, Lemma 7 is susceptible to slight improvements.

Now, we move the vertical integral path in (12) to the $35/54$ -line, picking up the residue of the integrand at 1. Thus

$$\begin{aligned} \int_{\theta-iT}^{\theta+iT} f_{\chi_1, \chi_2}(s) \frac{N^s}{s} ds &= \int_{\theta-iT}^{35/54-iT} f_{\chi_1, \chi_2}(s) \frac{N^s}{s} ds + \int_{35/54+iT}^{\theta+iT} f_{\chi_1, \chi_2}(s) \frac{N^s}{s} ds \\ &\quad + \int_{35/54-iT}^{35/54+iT} f_{\chi_1, \chi_2}(s) \frac{N^s}{s} ds + \text{Res}_{s=1} \left(f_{\chi_1, \chi_2}(s) \frac{N^s}{s} \right) \end{aligned}$$

Here, in order to deal with the contribution from the horizontal integrals, we cannot afford to use the crude convexity bounds on Dirichlet L -functions, due to the large number of L -functions in the integrand. Fortunately, a sharper bound has been established by Pan & Pan in Theorem 24.2.1 of [PP].

Lemma 8. Let $l \geq 3$, $L = 2^{l-1}$ and $\sigma_l = 1 - l(2L - 2)^{-1}$. Then when $\sigma \geq \sigma_l$

$$L(\sigma + it, \chi) \ll_a |t|^{1/(2L-2)} \log |t|$$

holds uniformly for $|t| \geq 2$.

When $l = 3$ we obtain

$$L(\sigma + it, \chi) \ll_a |t|^{1/6} \log |t|$$

uniformly for $|t| \geq 2$ and $\sigma \geq \frac{1}{2}$, and when $l = 4$,

$$L(\sigma + it, \chi) \ll_a |t|^{1/14} \log |t|$$

uniformly for $\sigma \geq 5/7$. Thus, by the convexity principle for Dirichlet series,

$$L(\sigma + it, \chi) \ll_a |t|^{\mu(\sigma) + \varepsilon}$$

uniformly for $|t| \geq 2$ and $\sigma \geq \frac{1}{2}$ where

$$\mu(\sigma) = \begin{cases} \frac{1}{6} - \frac{4}{9}(\sigma - \frac{1}{2}) & \text{when } \frac{1}{2} \leq \sigma \leq \frac{5}{7}, \\ \frac{1-\sigma}{4} & \text{when } \frac{5}{7} < \sigma \leq 1, \\ 0 & \text{when } 1 < \sigma. \end{cases}$$

We note that $\mu(\frac{35}{54}) = \frac{49}{486} < \frac{1}{9}$ and $\mu(\frac{5}{7}) = \frac{1}{14}$.

Now the horizontal paths contribute

$$\ll \int_{35/54}^{1+\varepsilon} N^\sigma |f_{\chi_1, \chi_2}(\sigma + iT)| T^{-1} d\sigma.$$

and this is

$$\ll \max_{35/54 \leq \sigma \leq 1+\varepsilon} N^\sigma T^{9\mu(\sigma)-1+\varepsilon},$$

and by the piecewise linearity of σ and $\mu(\sigma)$ this is

$$\ll N^{1+\varepsilon} T^{-1} + N^{5/7} T^{9\mu(5/7)-1+\varepsilon} + N^{35/54} T^{9\mu(35/54)-1+\varepsilon}.$$

When $T = N$ this is

$$\ll N^{35/54+\varepsilon}.$$

On the other hand, by Lemma 7 the vertical path also contributes

$$\ll N^{35/54+\varepsilon}.$$

The main term comes from the residual contributions, which, in the case that $\chi_1 = \chi_2 = \chi_0$, is $NP_8(\log N; a)$ where $P_8(\cdot; a)$ is a polynomial of degree 8 whose coefficients depend on a . Notice that for other choices of χ_1 and χ_2 , the residual contribution gives a polynomial of $\log N$ of lower degree than above.

For the leading coefficient, we need more precise information about f_{χ_0, χ_0} . By (4), (5) and (6) we have

$$f_{\chi_0, \chi_0} = L(s, \chi_0) \prod_{p \nmid a} \left(1 + \sum_{k=1}^{\infty} \frac{8k}{p^{ks}} \right) = L(s, \chi_0)^9 \prod_{p \nmid a} (1 + 6p^{-s} + p^{-2s})(1 - p^{-s})^6,$$

from which the leading coefficient is readily deduced. This completes the proof of Theorem 4.

In conclusion we remark that a concomitant argument will give

$$\sum_{\substack{n \leq N \\ (n, a) = 1}} R(n; a)^k = NP_{3^k-1}(\log N; a) + O_a(N^{\alpha_k + \varepsilon})$$

for any $\varepsilon > 0$, where $P_{3^k-1}(\cdot; a)$ is a polynomial of degree $3^k - 1$ whose coefficients depend on a and α_k is a constant that depends on the best 3^k -th power moment estimates for $L(s, \chi)$ in the critical strip and the quantity $\mu(\sigma)$ defined above. This question is closely related to the generalised divisor problem, and one is referred to Chapter 13 in Ivić [Iv] for more details.

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JJH: DEPARTMENT OF MATHEMATICS, MCALLISTER BUILDING, PENNSYLVANIA STATE UNIVERSITY, UNIVERSITY PARK, PA 16802-6401, U.S.A.

E-mail address: `huang@math.psu.edu`

RCV: DEPARTMENT OF MATHEMATICS, MCALLISTER BUILDING, PENNSYLVANIA STATE UNIVERSITY, UNIVERSITY PARK, PA 16802-6401, U.S.A.

E-mail address: `rvaughan@math.psu.edu`